ON UNIFORMLY BOUNDED BASIS IN SPACES OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. The main result of the paper is the construction of explicit uniformly bounded basis in the spaces of complex homogenous polynomials on the unit ball of C^3, extending an earlier result of the author in the C^2 case.

1. INTRODUCTION

This Note originates from the recent paper [S] that was kindly brought to the author's attention. In the introductory part of [S], the following two problems are put forward.

Problem 1. *Let B_d be the closed unit ball in $\mathbb{C}^d$ and $S_d = \partial B_d = \{\zeta \in \mathbb{C}^d;$ $\|\zeta\| = \langle\zeta, \zeta\rangle^{\frac{1}{2}} = 1\}$ the unit sphere. Denote for $N = 1, 2, 3\dots$*

$$\mathcal{P}_N = \mathcal{P}_N^{(d)} = \text{ span}\{\zeta^\alpha = \zeta_1^{\alpha_1}\cdots\zeta_d^{\alpha_d}; \alpha_i \geq 0 \text{ and } |\alpha| = \alpha_1+\cdots+\alpha_d = N\} \tag{1.1}$$

the space of degree N homogenous polynomials.
Do the spaces $\mathcal{P}_N$ have orthonormal basis that are uniformly bounded in $L^\infty(B_d)$? □

Problem 2. *Does the Hilbert space of homomorphic polynomials on S_d admit a uniformly bounded orthonormal basis? Same question for smooth strictly pseudo-convex domain $\Omega \subset \mathbb{C}^d$.* □

The first problem was solved affirmatively in [B] if $d = 2$, hence also answering Problem 2 for $d = 2$. Extending the approach from [B] to $d > 2$ turns out to be not straightforward. In this paper we will give an construction for $d = 3$ which potentially may be generalized to higher dimension, though this could require additional work. On the other hand, one can provide an affirmative solution to Problem 2, without going through Problem 1. The core of the argument is a general result on orthonormal basis, proven

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in [O-P] (and going back to a construction due to A. Olevskii, [Ol]), which seems little known outside the experts' circle.

2. Uniformly bounded orthonormal basis

We start with the following result (Theorem 2 in [O-P]).

Proposition 1. *Let E be a separable linear subspace of a Hilbert space $L^2(\mu)$, μ a probability measure. Then E admits an orthonormal basis consisting of uniformly bounded functions, if and only if*

(i) $E \cap L^\infty(\mu)$ *is dense in E in the $L^2(\mu)$-norm*
(ii) $E \cap \{f \in L^\infty(\mu) : \|f\|_\infty \leq 1\}$ *is not a totally bounded subset of* $L^2(\mu)$.

Let Ω be a smooth strictly pseudo-convex domain and $L^2(\mu) = L^2(\partial\Omega, \sigma)$ with σ the normalized surface measure of $\partial\Omega$. We take for E the restriction to $\partial\Omega$ of the linear space of holomorphic polynomials. Hence condition (i) is obviously satisfied. For $\Omega = B_d$, results from [R-W] and [K] provide a sequence of elements $p_N \in \mathcal{P}_N^{(d)}$ such that $\|p_N\|_2 = 1$ and $\|p_N\|_\infty = C_d$, taking care of condition (ii).

More generally, for $\Omega \subset \mathbb{C}^d$ smooth and strictly pseudo-convex, a result due to E. Low [Lo] asserts in particular that if $\phi > 0$ is a continuous function on $\partial\Omega$, then for all $\varepsilon > 0$, there exists $g \in A(\Omega)$ (the algebra of holomorphic functions on Ω that extend continuously to $\bar{\Omega}$) such that $|g| \leq \phi$ on $\partial\Omega$ and

$$\sigma(\{\zeta \in \partial\Omega; |g| \neq \phi\}) < \varepsilon. \tag{2.1}$$

Hence $A(\Omega) \cap \{f \in L^\infty(\partial\Omega); \|f\|_\infty \leq 1\}$ is not totally bounded in $L^2(\partial\Omega)$. Next, we are invoking a result of Henkin [H], Kerzman [K] and Lieb [Li] according to which elements of $A(\Omega)$ can be approximated uniformly on $\bar{\Omega}$ by functions holomorphic on a neighborhood of $\bar{\Omega}$, hence by holomorphic polynomials. Thus in conclusion, we again get condition (ii) satisfied. We proved

Proposition 2. *If $\Omega \subset \mathbb{C}^d$ is a smooth strictly pseudo-convex domain, then the holomorphic polynomials on $\partial\Omega$ admit a uniformly bounded orthonormal basis.*

3. Construction of uniformly bounded orthonormal basis in $\mathcal{P}_N^{(3)}$

Answering a question of W. Rudin, the author proved in [B] that for $d = 2$, the spaces $\mathcal{P}_N^{(2)}$ admit orthonormal basis that are uniformly bounded in $L^\infty(B_2)$. In this section, we revisit this construction, seeking for a higher dimensional extension and succeed in doing so for $d = 3$.

We believe that (unlike [B]) this approach may be generalizable and will indicate how.

Recall that for $d = 2$, the basis are explicit and simple to describe. More specifically, we introduce in [B] polynomials $\big(\zeta = (z, w)\big)$

$$\varphi_k(\zeta) = (N+1)^{-1/2} \sum_{j=0}^{N} \sigma_j \, e^{2\pi i \frac{jk}{N+1}} \, \frac{z^j w^{N-j}}{\|z^j w^{N-j}\|_2} \tag{3.1}$$

in $\mathcal{P}_N$, where $\{\sigma_j\}_{j=0}^N$ is a suitable unimodular sequence, which is taking to be the classical ± 1-valued Rudin-Shapiro sequence

$$\sigma_j = (-1)^{a_j} \ \text{ with } \ a_j = \sum \varepsilon_j \varepsilon_{i+1} \tag{3.2}$$

and ε_i the digits in the binary expansion of n.

Certainly, there are other choices since the only relevant property of $\{\sigma_j\}$ is bound

$$\max_\theta \Big| \sum_{j\in I} \sigma_j e(j\theta) \Big| \le C|I|^{\frac{1}{2}} \tag{3.3}$$

where $I \subset \mathbb{Z}$ is an arbitrary interval (we use the notation $e(\theta) = e^{2\pi i\theta}$).

Proposition 3. *The spaces* $\mathcal{P}_N^{(3)} = \text{span}[\zeta_1^{j_1}\zeta_2^{j_2}\zeta_3^{N-j_1-j_2}; j_1, j_2 \ge 0, j_1+j_2 \le N]$ *admit uniformly bounded basis.*

We need some notation. Let us assume N odd and define

$$\begin{aligned}
\Delta &= \{(j_1, j_2) \in \mathbb{Z}^2; j_1, j_2 \ge 0, j_1 + j_2 \le N\} \\
\Delta_0 &= \{(0, j); 0 \le j \le N\} \\
\Delta' &= \Big\{(j_1, j_2) \in \Delta; j_1 \ge 1, 0 \le j_2 \le \frac{N-1}{2}\Big\} \\
\Delta'' &= \Big\{(j_1, j_2) \in \Delta; j_1 \ge 1, \frac{N+1}{2} \le j_2 \le N-1\Big\}
\end{aligned}$$

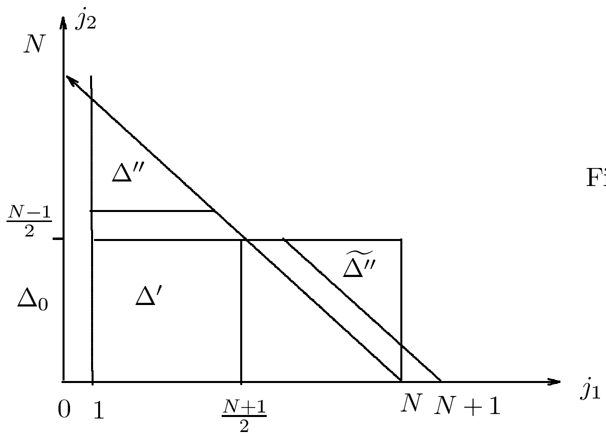

j_2
N
Δ''
$\frac{N-1}{2}$
$\widetilde{\Delta''}$
Δ_0
Δ'
j_1
0
1
$\frac{N+1}{2}$
N
$N+1$

Figure 1

Hence $\Delta = \Delta_0 \cup \Delta' \cup \Delta''$,

$$|\Delta| = \frac{1}{2}(N+1)(N+2) = D$$

$$|\Delta'| + |\Delta''| = \frac{N+1}{2} \cdot N.$$

Write the following orthogonal decomposition of $\mathcal{P}_N$

$$\mathcal{P}_N = X \oplus Y$$

with

$$X = \text{ span}\,[\zeta^\alpha; (\alpha_1, \alpha_2) \in \Delta' \cup \Delta'']$$

$$Y = \text{ span}\,[\zeta^\alpha, \alpha_1 = 0]$$

Invoking Proposition 1.4.9 from [R], if $\alpha = (\alpha_1, \dots, \alpha_d)$

$$\int_{\partial B_d} |\zeta^\alpha|^2 d\sigma = \frac{(d-1)!\,\alpha_1! \cdots \alpha_d!}{(d-1+\alpha_1+\cdots+\alpha_d)!}$$

and for $d = 3$

$$\|\zeta_1^{j_1}\zeta_2^{j_2}\zeta_3^{N-j_1-j_2}\|^2_{L^2(\partial B_3)} = \frac{2j_1!j_2!(N-j_1-j_2)!}{(N+2)!}. \tag{3.5}$$

Let us first consider the space

$$Y = [\zeta_2^{j_2}\zeta_3^{N-j_2}; 0 \le j_2 \le N].$$

Going back to (3.1), define for $k = 0, \dots, N$ the orthogonal system

$$\psi_k(\zeta) = (N+1)^{-\frac{1}{2}} \sum_{j_2=0}^{N} \sigma_j\, e\Big(\frac{j_2 k}{N+1}\Big)\, \frac{\zeta_2^{j_2}\zeta_3^{N-j_2}}{\|\zeta_2^{j_2}\zeta_3^{N-j_2}\|_{L^2(\partial B_3)}}.$$

Since by (3.4)

$$\|\zeta_2^{j_2}\zeta_3^{N-j_2}\|_{L^2(\partial B_3)} = \Big[\frac{2j_2!(N-j_2)!}{(N+2)!}\Big]^{\frac{1}{2}}$$

and

$$\|\zeta_2^{j_2}\zeta_3^{N-j_2}\|_{L^2(\partial B_2)} = \Big[\frac{j_2!(N-j_2)!}{(N+1)!}\Big]^{\frac{1}{2}}$$

it follows from the $d = 2$ construction that

$$\psi_k = \Big(\frac{N}{2}+1\Big)^{\frac{1}{2}} \varphi_k \tag{3.6}$$

with $\{\varphi_k\}$ a uniformly orthonormal basis on $L^2(\partial B_2)$.

In particular, we have

$$\|\psi_k\|_\infty \le cN^{\frac{1}{2}}. \tag{3.7}$$

Assume we constructed a uniformly bounded orthonormal basis $f_1, \dots, f_{N\frac{N+1}{2}}$ for the space X. One can then apply Olevskii's absorption scheme [Ol] to produce a uniformly bounded basis for $\mathcal{P}_N^{(3)}$. We recall the construction. Define for $k = 0, \dots, D-1$

$$g_k = a_{k,0}\,\psi_0 + \cdots + a_{k,N}\,\psi_N + a_{k,N+1}\,f_1 + \cdots + a_{k,D-1}\,f_{\frac{N(N+1)}{2}} \qquad (3.8)$$

where $A = (a_{k,\ell})_{0\le k,\ell<D} \in O(D)$ will be specified next.

Let

$$2^m \le D < 2^{m+1}$$

and

$$\begin{cases} a_{k,\ell} & = H^{(m)}(k,\ell) \ \text{ if } \ 0 \le k,\ell < 2^m \\ a_{k,\ell} & = 0 \ \text{ for } \ 0 \le k < 2^m, \ell \ge 2^m \\ a_{k,\ell} & = \delta_{k,\ell} \ \text{ for } \ 2^m \le k < D \end{cases} \qquad (3.9)$$

where $H^{(m)} = H$ is the discrete Haar system on $\{0, 1, \dots, 2^m - 1\}$. Thus

$$\begin{cases} H(e_0) & = \frac{1}{\sqrt{2^m}}\, 1_{[0,2^m[} \\ H(e_1) & = \frac{1}{\sqrt{2^m}}(1_{[0,2^{m-1}[} - 1_{[2^{m-1},2^m[}) \\ H(e_2) & = \frac{1}{\sqrt{2^{m-1}}}(1_{0,2^{m-2}[} - 1_{[2^{m-2},2^{m-1}[}) \\ H(e_3) & = \frac{1}{\sqrt{2^{m-1}}}(1_{[2^{m-1},2^{m-1}+2^{m-2}[} - 1_{[2^{m-1}+2^{m-2},2^m[}) \\ \text{etc.} & \end{cases} \qquad (3.10)$$

Clearly $A \subset \mathcal{O}(D)$. In view of (3.7)-(3.20), one easily verifies that

$$\begin{aligned} \|g_k\|_\infty &\le C\Big(\frac{1}{\sqrt{2^m}} + \frac{\sqrt{2}}{\sqrt{2^m}} + \cdots + \frac{(\sqrt{2})^{2\log N}}{\sqrt{2^m}}\Big)\sqrt{N} + C \\ &< C\frac{N}{\sqrt{D}} + C > C. \end{aligned}$$

Hence it remains to construct a uniformly basis for X.

Going back to Figure 1, let $\widetilde{\Delta''}$ be a triangle congruent to Δ'' with vertices at $\big(\frac{N+3}{2}, \frac{N-1}{2}\big)$, $(N,1)$, $\big(N, \frac{N-1}{2}\big)$ and let $T : \Delta'' \to \widetilde{\Delta''}$ be an affine map. For $k = (k_1, k_2) \in \Delta''$, denote $\tilde{k} = Tk$. Note that

$$\mathbb{Z}^2 \cap (\Delta' \cup \widetilde{\Delta''}) = \{1, \dots, N\} \times \Big\{0, \dots, \frac{N-1}{2}\Big\}. \qquad (3.11)$$

Define for $j=(j_1,j_2)\in\Delta'$

$$\eta_j=\frac{1}{\sqrt{\frac{N(N+1)}{2}}}\Big(\sum_{k\in\Delta'}u_k\,e\Big(\frac{j_1k_1}{N}+\frac{j_2k_2}{\frac{N+1}{2}}\Big)e_k+\sum_{k\in\Delta''}u_k\,e\Big(\frac{j_1\tilde{k}_1}{N}+\frac{j_2\tilde{k}_2}{\frac{N+1}{2}}\Big)e_k\Big) \tag{3.12}$$

and for $j\in\Delta''$

$$\eta_j=\frac{1}{\sqrt{N\frac{N+1}{2}}}\Big(\sum_{k\in\Delta'}u_k\,e\Big(\frac{\tilde{j}_1k_2}{N}+\frac{\tilde{j}_2k_2}{\frac{N+1}{2}}\Big)\,e_k+\sum_{k\in\Delta''}u_ke\Big(\frac{\tilde{j}_1\tilde{k}_i}{N}+\frac{\tilde{j}_2\tilde{k}_2}{\frac{N+1}{2}}\Big)\Big) \tag{3.13}$$

where

$$e_j=\frac{\zeta_1^{j_1}\zeta_2^{j_2}\zeta_3^{N-j_1-j_2}}{\|\zeta_1^{j_1}\zeta_2^{j_2}\zeta_3^{N-j_1-j_2}\|_{L^2(\partial B_3)}}$$

and $(u_k)=(u_{k_1k_2})$ will be some unimodular sequence.

We first verify that $(\eta_j)_{j\in\Delta'\cup\Delta''}$ is an orthonormal system.

By orthogonality and (3.11), if $j,j'\in\Delta'$

$$\begin{aligned}\langle\eta_j,\eta_{j'}\rangle&=\frac{1}{N\frac{N+1}{2}}\Big\{\sum_{k\in\Delta'}e\Big(\frac{j_1-j_1'}{N}k_1+\frac{j_2-j_2'}{\frac{N+1}{2}}k_2\Big)+\sum_{k\in\widetilde{\Delta''}}e\Big(\frac{j_1-j_1'}{N}k_1+\frac{j_2-j_2'}{\frac{N+1}{2}}k_2\Big)\Big\}\\&=\frac{1}{N\frac{N+1}{2}}\sum_{\substack{1\le k_1\le N\\0\le k_2\le\frac{N-1}{2}}}e\Big(\frac{j_1-j_1'}{N}k_1+\frac{j_2-j_2'}{\frac{N+1}{2}}k_2\Big)=\delta_{j,j'}\end{aligned}$$

and similarly for $j,j'\in\Delta''$.

For $j\in\Delta', j'\in\Delta''$, we obtain

$$\frac{1}{N\frac{N+1}{2}}\sum_{\substack{1\le k_1<N\\0\le k_2\le\frac{N-1}{2}}}e\Big(\frac{j_1-(\widetilde{j'})_1}{N}k_1+\frac{j_2-(\widetilde{j'})_2}{\frac{N+1}{2}}k_2\Big)=\delta_{j,\widetilde{j'}}=0.$$

Hence $(\eta_j)_{j\in\Delta'\cup\Delta''}$ is a basis for X.

Remains to introduce the sequence u_k. This is the main novel input compared with [B] (Rudin-Shapiro sequence based constructions do not seem to fit our purpose).

Define

$$u_{k_1,k_2}=e\big(\sqrt{2}(k_1^2+k_2^2)\big). \tag{3.14}$$

The only role of $\sqrt{2}$ is its diophantine property

$$\min_{x\in\mathbb{Z},x\neq0}|x|\,\|x\sqrt{2}\|>c>0 \tag{3.15}$$

($\| \ \|$ the distance to the nearest integer). Since $\sqrt{2}$ is a quadratic irrational, it has a periodic and hence bounded sequence of partial quotients, hence (3.15).

We will rely on the following two estimates, which also explain the role of (3.15).

Lemma 4. *Let I_1, I_2 be two arbitrary intervals of size M_1, M_2 and centers c_1, c_2. Rather than summing over $I_1 \times I_2$ we introduce a mollification, considering a smooth, symmetric, compactly supported bump function $0 \leq \rho \leq 1$ and a weight $\rho\big(\frac{k_1-c_1}{M_1}\big)\rho\big(\frac{k_2-c_2}{M_2}\big)$. The following inequalities hold*

$$\max_{\psi_1,\psi_2\in\mathbb{R}} \Big| \sum_k \rho\Big(\frac{k_1-c_1}{M_1}\Big)\rho\Big(\frac{k_2-c_2}{M_2}\Big) u_k \Big| \lesssim \sqrt{M_1 M_2} \tag{3.16}$$

and

$$\max_{\psi_1,\psi_2\in\mathbb{R}} \Big| \sum_k \rho\Big(\frac{k_1-c_1}{M_1}\Big)\rho\Big(\frac{k_2-c_2}{M_2}\Big) u_{k_1, N-k_1-k_2} \Big| \lesssim \sqrt{M_1 M_2}. \tag{3.17}$$

Proof.
$\boxed{3.16}$
Denoting $S = \sum_k \rho\big(\frac{k_1-c_1}{M_1}\big)\rho\big(\frac{k_2-c_2}{M_2}\big) e(k.\psi) u_k$, we obtain by squaring

$$\begin{aligned}
|S|^2 = \sum_{k,k'} & \rho\Big(\frac{k_1-c_1}{M_1}\Big)\rho\Big(\frac{k_1'-c_1}{M_1}\Big)\rho\Big(\frac{k_2-c_2}{M_2}\Big)\rho\Big(\frac{k_2'-c_2}{M_2}\Big) \\
& e\big((k-k').\psi\big)\, e\Big(\big(\sqrt{2}\,(k_1-k_1')(k_1+k_1') + (k_2-k_2')(k_2+k_2')\big)\Big) \\
= \sum_{k,k'} & \rho\Big(\frac{k_1}{M_1}\Big)\rho\Big(\frac{k_1'}{M_1}\Big)\rho\Big(\frac{k_2}{M_2}\Big)\rho\Big(\frac{k_2'}{M_2}\Big) e\big((k-k')\psi'\big) \\
& e\big(\sqrt{2}((k_1-k_1')(k_1+k_1') + (k_2-k_2')(k_2+k_2'))\big)
\end{aligned}$$

with $\psi' = \psi + 2\sqrt{2}c$. Making a change of variables $\ell = k - k'$, $\ell' = k + k'$, we obtain

$$\sum_{\ell,\ell'} \rho\Big(\frac{\ell_1+\ell_1'}{2M_1}\Big)\rho\Big(\frac{\ell_1-\ell_1'}{2M_1}\Big)\rho\Big(\frac{\ell_2-\ell_2'}{2M_2}\Big)\rho\Big(\frac{\ell_2-\ell_2'}{2M_2}\Big) e(\ell.\psi')\, e\big(\sqrt{2}(\ell_1\ell_1' + \ell_2\ell_2')\big)$$

which may be bounded by expressions of the form

$$\Big[\sum_{\ell_1,\ell_1'} \rho_1\Big(\frac{\ell_1}{L_1}\Big)\rho_2\Big(\frac{\ell_1'}{L_1'}\Big) e(\ell_1\psi_1')\, e(\sqrt{2}\ell_1\ell_1')\Big] \Big[\sum_{\ell_2,\ell_2'} \rho_1\Big(\frac{\ell_2}{L_2}\Big)\rho_2\Big(\frac{\ell_2'}{L_2'}\Big) e(\ell_2\psi_2') e(\sqrt{2}\ell_2\ell_2')\Big] \tag{3.18}$$

where $L_1, L_1' \lesssim M_1, L_2, L_2' \lesssim M_2$.

We estimate each of the factors of (3.18). We have

$$\Big|\sum_{\ell,\ell'} \rho_1\Big(\frac{\ell}{L}\Big)\rho_2\Big(\frac{\ell'}{L'}\Big)e(\ell\psi')e(\sqrt{2}\ell_1\ell_2')\Big| \leq \sum_{\ell} \rho_1\Big(\frac{\ell}{L}\Big)\Big|\sum_{\ell'} \rho_2\Big(\frac{\ell'}{L'}\Big)e(\sqrt{2}\ell\ell')\Big|. \tag{3.19}$$

By Poisson summation, the inner sum in (3.19) equals

$$\begin{aligned} &L' \sum_{k\in Z} \hat{\rho}_2\big(L'(k-\sqrt{2}\ell)\big) \leq \\ &L' \sum_{k\in\mathbb{Z}} \frac{1}{(L')^2|k-\sqrt{2}\ell|^2+1} \qquad \text{(since } \rho_2 \text{ is smooth)} \\ &\leq \frac{C}{L'} + \frac{1}{L'\|\sqrt{2}\ell\|^2+\frac{1}{L'}} \end{aligned}$$

and summation over ℓ gives

$$\frac{C}{L'}\sum_{\ell\lesssim L} \frac{1}{\|\sqrt{2}\ell\|^2+\frac{1}{(L')^2}}. \tag{3.20}$$

At this point, we use (3.15). Clearly (3.15) implies

$$|\{\ell \leq L; \|\sqrt{2}\ell\| \sim 2^{-s}\}\| \lesssim \frac{L}{2^s}+1$$

which permits to bound (3.20) by

$$\frac{C}{L'}\sum_{s, 2^s\leq L'} \Big(\frac{L}{2^s}+1\Big)4^s \lesssim C(L+L').$$

Hence (3.18) is bounded by $M_1.M_2$, proving (3.16).

$\boxed{(3.17)}$

Now

$$S = \sum_k \rho\Big(\frac{k_1-c_1}{M_1}\Big)\rho\Big(\frac{k_2-c_2}{M_2}\Big)\, e(k.\psi)\, e\big(\sqrt{2}\big((N-k_1-k_2)^2+k_1^2\big)\big)$$

hence

$$|S| = \Big|\sum_k \rho\Big(\frac{k_1-c_1}{M_1}\Big)\rho\Big(\frac{k_2-c_2}{M_2}\Big)\, e(k.\psi')\, e\big(\sqrt{2}\big((k_1+k_1)^2+k_1^2\big)\big)\Big|$$

for some $\psi' \in \mathbb{R}$. Proceeding as before, we obtain instead of (3.18) the following bound on $|S|^2$

$$\Big| \sum_{\ell_1,\ell_1',\ell_2,\ell_2'} \rho_1\Big(\frac{\ell_1}{L_1}\Big)\rho_2\Big(\frac{\ell_1'}{L_1'}\Big)\rho_1\Big(\frac{\ell_2}{L_2}\Big)\rho_2\Big(\frac{\ell_2'}{L_2'}\Big) e(\ell.\psi')\, e\Big(\sqrt{2}\big((\ell_1+\ell_2)(\ell_1'+\ell_2')+\ell_1\ell_1'\big)\Big)\Big|$$

$$\leq \sum_{\ell_1,\ell_2} \rho_1\Big(\frac{\ell_1}{L_1}\Big)\rho_1\Big(\frac{\ell_2}{L_2}\Big)\Big|\sum_{\ell_1'} \rho_2\big(\frac{\ell_1'}{L_1'}\big)\, e\big(\sqrt{2}(2\ell_1+\ell_2)\ell_1'\big)\Big|.\Big|\sum_{\ell_2'} \rho_2\Big(\frac{\ell_2'}{L_2'}\Big) e\big(\sqrt{2}(\ell_1+\ell_2)\ell_2'\big)\Big|$$

$$\leq \sum_{\substack{\ell_1\lesssim L_1\\ \ell_2\lesssim L_2}} \frac{1}{L_1'\|\sqrt{2}(2\ell_1+\ell_2)\|^2+\frac{1}{L_1'}}\cdot\frac{1}{L_2'\|\sqrt{2}(\ell_1+\ell_2)\|^2+\frac{1}{L_2'}}.$$

Assuming $L_1\geq L_2$, we obtain (performing first summation over ℓ_2)

$$\sum_{\substack{\ell_1\lesssim L_1\\ \ell_2\lesssim L_2}} \frac{1}{L_1'\|\sqrt{2}\ell_1\|^2+\frac{1}{L_1'}}\cdot\frac{1}{L_2'\|\sqrt{2}(2\ell_1-\ell_2)\|^2+\frac{1}{L_2'}}$$

$$< C(L_2+L_2')\sum_{\ell_1\lesssim L_1} \frac{1}{L_1'\|\sqrt{2}\ell_1\|^2+\frac{1}{L_1'}} < C(L_2+L_2')(L_1+L_1') < cM_1M_2$$

proving (3.17). □

The next distributional considerations are very similar to those in [B]. Fix $\zeta\in 0B_3$. We have by (3.5)

$$e_{k_1,k_2}(\zeta)=\frac{\zeta_1^{k_1}\zeta_2^{k_2}\zeta_3^{N-k_1-k_2}}{(2k_1!k_2!(N-k_1-k_2)!)^{\frac12}}\sqrt{(N+2)!}. \tag{3.21}$$

Let us assume $N-k_1-k_2\asymp N$. Otherwise, assuming say $k_2\asymp N$, we switch variables, writing $k_2=N-k_1-k_3$ and in this case

$$e_{k_1,k_3}(\zeta)=\frac{\zeta_1^{\ell_1}\zeta_3^{k_3}\zeta_2^{N-k_1-k_3}}{(2k_1!k_3!(N-k_1-k_3)!)^{\frac12}}\sqrt{(N+2)!}. \tag{3.22}$$

Writing

$$(3.21)=e\big(k_1\psi_1+k_2\psi_2+(N-k_1-k_3)\psi_3\big)\,\frac{|\zeta_1|^{k_1}|\zeta_2|^{k_2}(1-\zeta_1^2-\zeta_2^2)^{\frac{N-k_1-k_2}{2}}}{(2k_1!k_2!(N-k_1-k_2)!)^{\frac12}}\sqrt{(N+2)!}$$

for some ψ_1,ψ_2,ψ_3 (note that this first factor is harmless in view of the formulation of Lemma 4), we first need to analyze the distribution of

$$\frac{|\zeta_1|^{k_1}|\zeta_2|^{k_2}(1-\zeta_1^2-\zeta_2^2)^{\frac{N-k_1-k_2}{2}}}{(k_1!k_2!(N-k_1-k_2)!)^{\frac12}}\sqrt{(N+2)!}. \tag{3.23}$$

Set $t_1 = |\zeta_1|^2, t_2 = |\zeta_2|^2$. By Stirling's formula

$$(3.23) \sim \frac{t_1^{k_{1/2}} t_2^{k_{2/2}} (1-t_1-t_2)^{\frac{N-k_1-k_2}{2}} (N+2)^{\frac{N+2}{2}} (N+2)^{\frac{1}{4}}}{k_1^{1/2} k_2^{1/2} (N-k_1-k_2)^{\frac{N-k_1-k_2}{2}} k_1^{\frac{1}{4}} k_2^{\frac{1}{4}} (N-k_1-k_2)^{\frac{1}{4}}}$$

$$\sim \frac{N.N^{\frac{1}{4}}}{k_1^{1/4} k_2^{1/4} (N-k_1-k_2)^{\frac{1}{4}}} \Big(\frac{t_1}{k_1 N^{-1}}\Big)^{k_{1/2}} \Big(\frac{t_2}{k_2 N^{-1}}\Big)^{k_{2/2}} \Big(\frac{1-t_1-t_2}{1-k_1N^{-1}-k_2N^{-1}}\Big)^{\frac{N-k_1-k_2}{2}} \tag{3.24}$$

and because of the normalization factor $\frac{1}{\sqrt{N^{\frac{N+1}{2}}}}$ in (3.12), (3.13), we may drop the N factor in (3.24).

Write for $u = t + \Delta u$, $\Delta u = o(t)$

$$\Big(\frac{t}{u}\Big)^u = e^{-(t+\Delta u)(\frac{\Delta u}{t} - \frac{1}{2}(\frac{\Delta u}{t})^2 + \cdots)} = e^{-\Delta u - \frac{(\Delta u)^2}{2t} + \cdots}$$

Hence (3.24) gives (after removal of the N-factor)

$$\frac{N^{\frac{1}{4}}}{k_1^{\frac{1}{4}} k_2^{\frac{1}{4}} (N-k_1-k_2)^{\frac{1}{4}}} e^{-\frac{N}{2}[\frac{(k_1N^{-1}-t_1)^2}{t_1} + \frac{(k_2N^{-1}-t_2)^2}{t_2} + \frac{((k_1+k_2)N^{-1}-t_1-t_2)^2}{1-t_1-t_2} + \cdots]} \tag{3.25}$$

and the distribution in (k_1, k_2)-space localizes to

$$\begin{cases} |k_1 - t_1N| \lesssim \sqrt{t_1N} \\ |k_2 - t_2N| \lesssim \sqrt{t_2N}. \end{cases} \tag{3.26}$$

Thus, if we fix a center $\bar{k} = ([t_1N], [t_2N])$ (3.26) corresponds to the tile

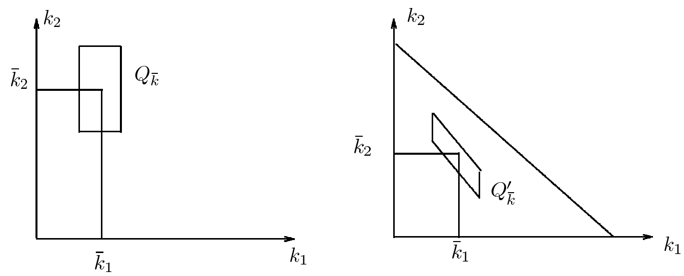
$$Q_k = \{k \in \Delta' \cup \Delta''; |k_1 - \bar{k}_1| \lesssim \sqrt{\bar{k}_1}, |k_2 - \bar{k}_2| \lesssim \sqrt{\bar{k}_2}\}. \tag{3.27}$$

Note that for (3.22), we obtain a tile with a different shape

$$Q'_{\bar{k}} = \{k \in \Delta' \cup \Delta''; |k_1 - \bar{k}_1| \lesssim \sqrt{\bar{k}_1}; |k_1 + k_2 - \bar{k}_1 - \bar{k}_2| \lesssim \sqrt{N - \bar{k}_1 - \bar{k}_2}\} \tag{3.28}$$

or with

$$|k_1 - \bar{k}_1| \lesssim \sqrt{\bar{k}_1} \text{ replaced by } |k_2 - \bar{k}_2| \lesssim \sqrt{\bar{k}_2}. \tag{3.29}$$



Going back to (3.12), (3.13), some[1] attention is required due to the presence of the different factors

$$e\Big(j_1\frac{k_1}{N}+j_2\frac{k_2}{N}\Big) \text{ and } e\Big(j_1\frac{\tilde{k}_1}{N}+j_2\frac{\tilde{k}_2}{N}\Big)$$

depending on whether $k\in\Delta'$ or $k\in\Delta''$. Each argument is affine in k, but with a different expression. Hence it is natural to apply Lemma 4 to the

intersections

$$Q_k \cap \Delta' \quad Q_k \cap \Delta'' \tag{3.30}$$

$$Q'_k \cap \Delta' \quad Q'_k \cap \Delta''. \tag{3.31}$$

The Q_k appear when $N - k_1 - k_2 \asymp N$ and hence $Q_k \cap \Delta'$, $Q_k \cap \Delta''$ are still boxes.

If $Q'_k \cap \Delta' \neq \phi$, $Q'_k \cap \Delta'' \neq \phi$ (assuming, as we may, that $N - k_1 - k_2 < \frac{N}{100}$), clearly $k_1 \approx \frac{N}{2}, k_2 \approx \frac{N}{2}$ and Q'_k has length $\sim \sqrt{N}$. Thus we may then in either case (3.28), (3.29) take for $Q'_{\bar{k}}$ a box

$$Q'_{\bar{k}} = \{(k_1, k_2) \in \Delta' \cup \Delta''; |k_2 - \bar{k}_2| \lesssim \sqrt{N} \text{ and } |k_1 + k_2 - \bar{k}_1 - \bar{k}_2| \lesssim \sqrt{N - \bar{k}_1 - \bar{k}_2}\}.$$

The intersections $Q'_{\bar{k}} \cap \Delta', Q''_{\bar{k}} \cap \Delta''$ have the same structure, i.e. k_2 and $k_1 + k_2$ restricted to suitable intervals.

Applying Lemma 4 to the summation for $k \in Q_{\bar{k}} \cap \Delta'$, $k \in Q_{\bar{k}} \cap \Delta''$ (after proper mollification as required in Lemma 4) gives the bound

$$C \frac{N^{\frac{1}{4}}}{\bar{k}_1^{\frac{1}{4}} \bar{k}_2^{\frac{1}{4}} (N - \bar{k}_1 - \bar{k}_2)^{\frac{1}{4}}} (\bar{k}_1 \bar{k}_2)^{\frac{1}{4}} < C$$

and for $k \in Q'_{\bar{k}} \cap \Delta', k \in Q'_{\bar{k}} \cap \Delta''$ (assuming $\bar{k}_2 \asymp N$)

$$C \frac{N^{\frac{1}{4}}}{\bar{k}_1^{\frac{1}{4}} \bar{k}_2^{\frac{1}{4}} (N - \bar{k}_1 - \bar{k}'_2)^{\frac{1}{4}}} \big(\bar{k}_1 (N - \bar{k}_1 - \bar{k}_2)\big)^{\frac{1}{4}} < C.$$

Similarly to [B], we perform a tiling of $\Delta' \cup \Delta''$ as dictated by (3.25) and exploit the exponentially decaying factors to get a bounded collected contribution (the reader will easily check details). This completes the proof of Proposition 3.

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